Exercise 3.1. Let V be an *n*-dimensional complex vector space and *L* a matrix. *L* has at least one eigenvalue, and each distinct eigenvalue has a corresponding (non-zero) eigenvector. In particular, *L* has at least one eigenvector. If *L* is Hermitian then V has an orthonormal basis consisting of eigenvectors of *L*.

Proof. Let .  is clearly a polynomial in **, and it is known as the characteristic polynomial of *L*. By the Fundamental Theorem of Algebra, there are complex roots  of  of multiplicity *pi* such that  where *r* is a positive integer and . In particular, there is at least one , and for all *i*, . For each *i*, by footnote (\*), there is a non-zero vector  such that . That is,  is an eigenvalue of *L* with eigenvector . This proves that every matrix *L* has at least one non-zero eigenvector and that each distinct eigenvalue has a corresponding (non-zero) eigenvector.

Now suppose that *L* is Hermitian. Then all eigenvalues  are real. WLOG we can assume  is a unit vector, . Define the null space . It is easy to see that *N* is a vector subspace of *L*. Since dim  is a 1‑dimensional subspace, the orthogonal subspace *N* has dimension *n* – 1. Claim *LN N*:

Let . We need to show that . Since *L* is Hermitian, . So, we need to show that :

 ✔

Let  restricted to *N*. Repeating our logic above,  has a real root  that is an eigenvalue of  with corresponding unit eigenvector . Since ,  .

Using the (*n* – 2)-dimensional null space of  as above we generate , and since  also.

Continuing this process, we eventually obtain the orthonormal basis . ■

Footnote (\*)

First, suppose we have *n* equations in *n* unknowns:



By Cramer’s Rule, det *A* ≠ 0 ⇔ there exists a unique vector such that  Since is the unique solution. That is, det A ≠ 0 if and only if  is the unique solution to 

So, if det *A* = 0, then while  is a solution it is not unique. That is, there is an  such that .

Now we apply this fact to the characteristic polynomial. Fix *i* and let . Then

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Define the vector  Then

.

That is, for each  there is a vector  such that .