Exercise 3.1 Let V be an *n*-dimensional complex vector space. If *L* is a Hermitian matrix then V has an orthonormal basis consisting of eigenvectors of *L*.

Proof. First we must show that *L* ***has*** an eigenvector. Then we will show how to extend it to a full basis. For this first step, *L* can be any matrix, not necessarily Hermitian.

 is a polynomial in ** known as the characteristic polynomial of *L*. By the Fundamental Theorem of Algebra, there are complex roots  of  of multiplicity *pi* such that  where *r* is a positive integer and . In particular,  has at least one root,.

By the Lemma, below, there is an eigenvector  having  as an eigenvalue. This proves that every matrix *L* has at least one non-zero eigenvector. WLOG we can assume  is a unit vector because we can, if necessary, divide  by its magnitude and it will still be an eigenvector having  as its eigenvalue.

Define the null space . It is easy to see that N is a vector subspace of V.

Claim dim N = *n* – 1:

 can be extended to a basis of V, and the sub-basis vectors belong to N because  for all *i*. ✔

Claim *L*N** N:

Let . We need to show that . Since *L* is Hermitian, *L* = , the transpose of the complex conjugate of *L*. Thus, . So, we need to show that :

 ✔

Let  restricted to N. Repeating our logic above,  has a root that is an eigenvalue of  with corresponding unit eigenvector . Since ,  .

Restricting *L* to the (*n* – 2)-dimensional null space of , as above we generate unit eigenvector  such that , and since  also.

Continuing this process, we eventually obtain the orthonormal basis . ■

Lemma Every real or complex matrix *A* has at least one (possibly complex) eigenvector corresponding to the root  of the characteristic polynomial.

Proof. We seek a non-zero vector  that satisfies . Let  be an orthonormal basis for V. Every matrix *A* =  represents a linear transformation  defined by  on the basis vectors and then extended linearly to all of *V*.

A linear transformation *T* is **singular** if dim *TV* < *n*, and *T* is singular iff det *A* ≠ 0.

Set . Since  = 0, *B* is singular. That means that the *n*-equations in *n*-unknowns,



has redundant rows. In theory, one or more rows can be eliminated by row reduction. Specifically, let *m* = dim *TV*. Assuming *A* ≠ 0, then 0 < *m* < *n*, and so

*n* – *m* unknowns are redundant and thus are free variables that can be given any value. We set the free variables equal to 1 and then solve the remaining equations for the unique values of . Then  is a non-zero vector that satisfies

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That is, we have found an eigenvector having  as an eigenvalue. ■