Exercise 3.1 . Let V be an *n*-dimensional complex vector space. If *L* is Hermitian then V has an orthonormal basis consisting of eigenvectors of *L*.

Proof. First we must show that *L* ***has*** an eigenvector. Then we will show how to extend it to a full basis. For this first step, *L* can be any matrix, not necessarily Hermitian.

 is a polynomial in ** known as the characteristic polynomial of *L*. By the Fundamental Theorem of Algebra, there are complex roots  of  of multiplicity *pi* such that  where *r* is a positive integer and . In particular,  has at least one root,.

By footnote (\*), below, there is an eigenvector  having  as an eigenvalue. This proves that every matrix *L* has at least one non-zero eigenvector. WLOG we can assume  is a unit vector because we can, if necessary, divide  by its magnitude and it will still be an eigenvector having  as its eigenvalue.

Define the null space . It is easy to see that N is a vector subspace of V.

Claim dim N = *n* – 1:

 can be extended to a basis of V, and the sub-basis vectors belong to N because  for all *i*. ✔

Claim *L*N** N:

Let . We need to show that . Since *L* is Hermitian, . So, we need to show that :

 ✔

Let  restricted to N. Repeating our logic above,  has a root that is an eigenvalue of  with corresponding unit eigenvector . Since ,  .

Restricting *L* to the (*n* – 2)-dimensional null space of , as above we generate unit eigenvector  such that , and since  also.

Continuing this process, we eventually obtain the orthonormal basis . ■

**Footnote (\*)**

First, suppose we have *n* equations in *n* unknowns:



By Cramer’s Rule, det *A* ≠ 0 ⇔ there exists a unique vector such that  So, if det *A* = 0, then while  is a solution, it is not unique. That is, there is an  such that .

Now we apply this fact to the characteristic polynomial of matrix *L*. Let . Since  is a root of ,

 



So, , or . That is,  is an eigenvector of *L*, and we denote it as . Thus, we have found an eigenvector  having  as an eigenvalue. ✔