Exercise 3.1 Let V be an *n*-dimensional complex vector space. If *L* is a Hermitian matrix then V has an orthonormal basis consisting of eigenvectors of *L*.

Proof. First we must show that *L* ***has*** an eigenvector. Then we will show how to extend it to a full basis. For this first step, *L* can be any matrix, not necessarily Hermitian.

 is a polynomial in ** known as the characteristic polynomial of *L*. By the Fundamental Theorem of Algebra, there are complex roots  of  of multiplicity *pi* such that  where *r* is a positive integer and . In particular,  has at least one root,.

By the Lemma, below, there is an eigenvector  having  as an eigenvalue. This proves that every matrix *L* has at least one non-zero eigenvector. WLOG we can assume  is a unit vector because we can, if necessary, divide  by its magnitude and it will still be an eigenvector having  as its eigenvalue. Define the null space of :

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It is easy to see that N1 is a vector subspace of V. Claim dim N1 = *n* – 1:

Using the Gram-Schmidt Orthogonalization process,  can be extended to an orthonormal basis of V, and the basis vectors belong to N1 because  for all *i*. ✔

Claim *L*N1** N1:

Let . Let . We need to show that . Since

 and *L* is Hermitian (*L* = ),   ✔

In the Lemma, below, we showed that *L* is *associated* with a linear transformation *T* on V. Let *T*2 be the linear transformation generated by restricting *T* to N1, the (*n* – 1) dimensional null space of , and let *L*2 be the matrix associated with *T*2. Repeating our logic above,  has a root that is an eigenvalue of  with corresponding unit eigenvector . Since ,  .

Let *T*3 be the linear transformation generated by restricting *T*2 to N2, the (*n* – 2) dimensional null space of , and let *L*3 be the matrix associated with *T*3. As above, we generate unit eigenvector  such that , and since  also. Continuing this process, we eventually obtain the orthonormal basis . ■

Lemma Every real or complex matrix *A* has at least one (possibly complex) eigenvector corresponding to the root  of the characteristic polynomial.

Proof. We seek a non-zero vector  that satisfies . Let  be an orthonormal basis for V. Every matrix *A* =  is **associated** with a linear transformation  defined by  on the basis vectors and then extended linearly to all of *V*.

A linear transformation *T* is **singular** if dim *TV* < *n*, and *T* is singular iff det *A* ≠ 0.

Set . Since  = 0, *B* is singular, and *T* maps *V* into an *m* dimensional subspace, where *m* = dim *TV* < *n*. The set of *n* linear equations in *n* unknowns,



has *n* – *m* redundant rows that can be eliminated by row reduction. There are then *n* – *m* free variables that can be given any value. We set the free variables equal to 1 (i.e., non-zero) and then solve the remaining equations for the unique values of . The resulting solution  is a non-zero vector that satisfies

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That is, we have found an eigenvector having  as an eigenvalue. ■

Note: We have proven that the complex characteristic root  is in fact an eigenvalue. Since *L* is Hermitian, then  is in fact real. If *L*, in addition, has real elements, we just call it symmetric (rather than Hermitian) and the eigenvectors, also, are real because  constitutes a system of real linear equations that can be solved using just addition, subtraction, multiplication, and division.